Kinetic description for a suspension of inelastic spheres - Boltzmann and BGK equations

Cédric Croizet and Renée Gatignol

Laboratoire de Modélisation en Mécanique - Université Pierre et Marie Curie (Paris 6) et CNRS UMR 7607 - 4, place Jussieu - 75252 Paris cedex 05 - France

Abstract. The problem of a two-phase dispersed medium is studied within the kinetic theory. In the case of small and undeformable spheres that have all the same radius, a model for the collision operator of the Boltzmann equation is proposed. The collisions are supposed instantaneous, binary and inelastic. The obtained collision operator allows, to prove the existence of an H theorem in several configurations according to the assumptions made about the particles and particularly in the case of a diluted suspension of weakly inelastic collisions. Because of the complexity of the non linear structure of the collision integral, the Boltzmann equation is very difficult to solve and to analyse. It is therefore interesting to introduce a BGK model equation to study qualitatively its solution. In order to be physically realistic and consistent with the assumptions related to the collisions, a collision frequency depending on the particles velocities is chosen. Moreover, the collision frequency is expected to vary strongly with the particles velocities. Taking this into account, a BGK model is written. All the basic properties of the original operator are retained.

I BOLTZMANN EQUATION

The problem of a two-phase dispersed medium is studied within the kinetic theory. In the case of small and undeformable spheres that have all the same radius $(D/2)$, a model for the collision operator of the Boltzmann equation is proposed. Firstly, we consider the binary, inelastic and instantaneous collision of two identical hard spheres of radius $D/2$ (P_1 and P_2). Before the collision we suppose that the particle P_1 is centered at position $\vec{x_1}$ and has the velocity $\vec{\xi_1}$. The particle P_2 is centered at position $\vec{x_2}$ and has the velocity $\vec{\xi_2}$. So, the relative velocity is $\vec{g} = \vec{\xi_2} - \vec{\xi_1}$ and the impact vector is $\vec{k} = (\vec{x_1} - \vec{x_2})/ \parallel \vec{x_1} - \vec{x_2} \parallel$. After the collision, we suppose that the particle P_1 is centered at position $\vec{x_1}$ and has the velocity $\vec{\xi_1}\,'$. The particle P_2 is centered at position $\vec{x_2}$ and has the velocity $\vec{\xi_2}\,'$, and the relative velocity after collision is $\vec{g}\,' = \vec{\xi_2}\,' - \vec{\xi_1}\,'$. Furthermore, because of the inelasticity of the collision, we introduce the coefficient of restitution e. It is defined by the following relation between the components of the relative velocities normal to the plane of contact.

$$(\vec{g} \cdot \vec{k}) = -e(\vec{g}\,' \cdot \vec{k}) \tag{1}$$

If $e = 1$ the energy is conserved during the choc and if $e < 1$ energy is dissipated during a collision. By considering the momentum balance and the previous assumption, we can express the particle velocities just after the collision in term of those just before and reciprocally :

$$\vec{\xi_1}\,' = \vec{\xi_1} + \frac{1+e}{2}\left[\left(\vec{g} \cdot \vec{k}\right)\vec{k}\right] \tag{2}$$

$$\vec{\xi_2}\,' = \vec{\xi_2} - \frac{1+e}{2}\left[\left(\vec{g} \cdot \vec{k}\right)\vec{k}\right] \tag{3}$$

$$\vec{\xi_1} = \vec{\xi_1}\,' + \frac{1+e}{2e}\left[\left(\vec{g}\,' \cdot \vec{k}\right)\vec{k}\right] \tag{4}$$

$$\vec{\xi_2} = \vec{\xi_2}\,' - \frac{1+e}{2e}\left[\left(\vec{g}\,' \cdot \vec{k}\right)\vec{k}\right] \tag{5}$$

We consider the following Boltzmann equation (6) for a suspension.

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Abstract

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$$\frac{\partial f}{\partial t} + \vec{\xi_1} \cdot \frac{\partial f}{\partial \vec{x}_1} + \vec{F} \cdot \frac{\partial f}{\partial \vec{\xi_1}} = C(f,f) \tag{6}$$

where $\vec{F}$ is the external force (due particularly to the carrying fluid) acting on a particle and where we have introduced the distribution fonction $f(\vec{\xi},\vec{x},t)$ which is defined so that $f(\vec{\xi},\vec{x},t)d\vec{\xi}$ is the probable number of particules in $\vec{x}$, at time t with velocities in the volume element $d\vec{\xi}$ centered in $\vec{\xi}$ and denoted by $(\vec{\xi}, d\vec{\xi})$. We introduce also the distribution function $f^{(2)}$ which characterizes the statistic of binary collisions and depends on the velocities and positions of two particules and of time. It is defined so that

$$f^{(2)}(\vec{\xi_1},\vec{x_1},\vec{\xi_2},\vec{x_2},t)d\vec{\xi_1}d\vec{\xi_2}d\vec{x_1}d\vec{x_2} \tag{7}$$

is the probable number of pairs of particles which, at time t, are in $(\vec{x_1}, d\vec{x_1})$ and $(\vec{x_2}, d\vec{x_2})$ with velocities respectively in $(\vec{\xi_1}, d\vec{\xi_1})$ and $(\vec{\xi_2}, d\vec{\xi_2})$. We have to determine $C(f,f)$ in term of $f^{(2)}$. Using a very classical method ([4]) in kinetic theory based on the change in $(\vec{x_1}, d\vec{x_1})$ of the number of particles with velocity in $(\vec{\xi_1}, d\vec{\xi_1})$ during dt, we obtain ([5]) the following expression for the collision operator of the Boltzmann equation :

$$\begin{aligned} \frac{\partial f}{\partial t} + \vec{\xi_1} \cdot \frac{\partial f}{\partial \vec{x_1}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{\xi_1}} = \frac{D^2}{e^2} \iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k}) \Big\{ & f^{(2)}(\vec{\xi_1}', \vec{x_1}, \vec{\xi_2}', \vec{x_1} + D\vec{k}, t) \\ & - e^2 f^{(2)}(\vec{\xi_1}, \vec{x_1}, \vec{\xi_2}, \vec{x_1} - D\vec{k}, t) \Big\} d\vec{k} d\vec{\xi_2} \end{aligned} \tag{8}$$

We choose for $f^{(2)}$ the formulation given by [4] :

$$f^{(2)}(\vec{\xi_1},\vec{x_1},\vec{\xi_2},\vec{x_1}+D\vec{k},t) = g_o(\vec{x_1} + D\vec{k}/2) f(\vec{\xi_1},\vec{x_1}) f(\vec{\xi_2},\vec{x_1}+D\vec{k}) \tag{9}$$

It allows to take into account the dimension of the particles. This expression introduces the function $g_o(\vec{x})$ which expresses the influence of the density of the cloud of particles upon the probability of collision : $g_o(\vec{x}) = 1$ for a diluted suspension and $g_o(\vec{x})$ becomes infinite when the density is so great that the suspension behaves like a solid. Particular cases allow the validation of the operator. With $g_o(\vec{x}) = 1$, $e = 1$ and $\vec{x}_1 = \vec{x}_2$ (punctual collisions) the collision operator of the usual kinetic theory ([3], [4], [10]) is found. If we suppose that $g_o(\vec{x}) = 1$ and $\vec{x}_1 = \vec{x}_2$ the operator of [14] is obtained. At last, with $e = 1$, the Enskog equation ([4] and [2]) is obtained.

The function $\psi(\vec{\xi_1}, \vec{x_1})$ is introduced and the equation of moments is written in the classical way of the kinetic theory. We put $\psi(\vec{\xi_i}, \vec{x_1}) = \psi_i$ with $i = 1, 2$ and $\psi(\vec{\xi_i}', \vec{x_1}) = \psi_i'$ with $i = 1, 2$. It has been seen in [5] that the contribution of collisions $\mathcal{C}(\psi)$ to the transport equation can be written in the same form as the expression directly postulated by [9] and [4] (for dense gas) without introducing the Boltzmann equation :

$$\begin{aligned} \mathcal{C}(\psi) = \frac{D^2}{2} \iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k}) \Big[& (\psi_1' - \psi_1) f^{(2)}(\vec{\xi_1}, \vec{x_1}, \vec{\xi_2}, \vec{x_1} - D\vec{k}, t) \\ & + (\psi_2' - \psi_2) f^{(2)}(\vec{\xi_2}, \vec{x_1}, \vec{\xi_1}, \vec{x_1} + D\vec{k}, t) \Big] d\vec{k} d\vec{\xi_2} d\vec{\xi_1} \end{aligned} \tag{10}$$

with (2) and (3). This expression provide the conservation of mass ($\mathcal{C}(1) = 0$). If we consider, in addition to the initial assumptions, that the diameter of the particles is very small compared to the characteristic length (L) of the flow, and thanks to an asymptotic expansion in terms of the small parameter $D/L << 1$, we obtain a relation that allows to prove the conservation of momentum : the collision term of the momentum balance equation may be written under the form of a divergence term ([9]). Assuming that e is nearly one (weakly inelastic collisions) and using a linear combination, we can express $\mathcal{C}(\psi)$ in a rather symetrical form [5] :

$$\begin{aligned} \mathcal{C}(\psi) = \frac{-D^2}{2(1+e^2)} \iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k}) \Big[& \Big(\psi_1' - \psi_1\Big) \Big(f^{(2)}(\vec{\xi_1}', \vec{x_1}, \vec{\xi_2}', \vec{x_1} + D\vec{k}, t) \\ & - f^{(2)}(\vec{\xi_1}, \vec{x_1}, \vec{\xi_2}, \vec{x_1} - D\vec{k}, t) \Big) + (\psi_2' - \psi_2) \Big(f^{(2)}(\vec{\xi_2}', \vec{x_1}, \vec{\xi_1}', \vec{x_1} - D\vec{k}, t) \\ & - f^{(2)}(\vec{\xi_2}, \vec{x_1}, \vec{\xi_1}, \vec{x_1} + D\vec{k}, t) \Big) \Big] d\vec{k} d\vec{\xi_2} d\vec{\xi_1} \end{aligned} \tag{11}$$

with (2) and (3). Because of the exchange of energy between the particles and the carrying fluid, this expression of $\mathcal{C}(\phi)$ does not provide the conservation of this quantity. If we consider, in addition to the initial assumptions, that the diameter of the particles is very small compared to the characteristic length (L) of the flow, and thanks to an asymptotic expansion in terms of the small parameter $D/L << 1$, we obtain a relation that allows to prove the H-theorem.

The functions $\mathcal{H}(\vec{x_1},t)$ and $\vec{\phi}_H(\vec{x_1},t)$ are defined by the relations (12) and (13) respectively :

$$\mathcal{H}(\vec{x_1},t) = \int f(\vec{\xi_1},\vec{x_1},t) ln \left[f(\vec{\xi_1},\vec{x_1},t)\right] d\vec{\xi_1} \tag{12}$$

$$\vec{\phi}_H(\vec{x_1},t) = \int \vec{\xi_1} f(\vec{\xi_1},\vec{x_1},t) ln \left[f(\vec{\xi_1},\vec{x_1},t)\right] d\vec{\xi_1} \tag{13}$$

Classically the relation (14) is obtained.

$$\frac{\partial}{\partial t}\mathcal{H}(\vec{x_1},t) + \frac{\partial}{\partial \vec{x_1}} \cdot \vec{\phi}_H(\vec{x_1},t) = \mathcal{C}(ln(f(\vec{\xi_1},\vec{x_1},t)) \tag{14}$$

If we consider, in addition to the initial assumptions, that the diameter of the particles is very small compared to the characteristic length (L) of the flow, and thanks to an asymptotic development of (9) in terms of the small parameter $D/L << 1$, we obtain the following relation :

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{H}(\vec{x_1},t) + \frac{\partial}{\partial \vec{x_1}}\vec{\phi}_H(\vec{x_1},t) = \frac{-D^2e^2}{2(1+e^2)} \iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k}) \Big(f(\vec{\xi_1}',\vec{x_1}) f(\vec{\xi_2}',\vec{x_1}) \\ - f(\vec{\xi_1},\vec{x_1}) f(\vec{\xi_2},\vec{x_1}) \Big) ln \left(\frac{f(\vec{\xi_1}',\vec{x_1}) f(\vec{\xi_2}',\vec{x_1})}{f(\vec{\xi_1},\vec{x_1}) f(\vec{\xi_2},\vec{x_1})} \right) d\vec{k} d\vec{\xi_2} d\vec{\xi_1} + \mathcal{O}(D^3)\end{aligned} \tag{15}$$

Easily, we proove that the first order of the previous expression is lower than zero. It means that the quantity $\partial\mathcal{H}/\partial t + (\partial./\partial\vec{x_1}).\vec{\phi}_H$ is lower than zero if the particles are sufficiently small. Consequently, the H-theorem is true for small particles compared to the characteristic length (L) of the flow. In the case of punctual collisions, the balance law of the $\mathcal{H}$ quantity is reduced to the first order of the previous expression. Then, we are in the same conditions as [14] but we assume additionaly that the collisions are weakly inelastic. It allows to prove that $\partial\mathcal{H}/\partial t + (\partial./\partial\vec{x_1}).\vec{\phi}_H < 0$ and therefore that the H-theorem is true in that case. These results can be generalized to the case of a bounded medium subject to the choice of the boundary conditions.

II BGK MODEL

Because of the complexity of the non linear structure of the collision integral, the Boltzmann equation is very difficult to solve and to analyse. It is therefore interesting to introduce a simplified model equation to study qualitatively its solution. There are principally two kinds of methods to obtain these models from the Boltzmann equation. The first consists in a linearization of the Boltzmann equation. We find this approach in [8] for instance. The second is based on the unlinearized Boltzmann equation. The obtained model equation has several properties of the original operator, but it is not strictly derived from it. We find this approach in [1], [12], [11] for instance. An approach by a method of the first category seems to be difficult to apply in the case of not punctual collisions of inelastic spheres, studied here. Indeed, the use of a method of linearization, requires, for the construction of the model, the conservation of mass, momentum and energy which is not guaranteed by the original model. In this work, an approach by the second method, which has not the same requirements as for these properties, will be retained. The method introduced in [12] seems to be difficult to implement in the case studied here. Indeed, it requires the choice of a model distribution function f_M which allows to impose constraints on model and which is the limit distribution function when the Knudsen number K_n of the flow aims towards 0. The choice of this function is complicated enough. For instance, the distribution function of "13 moments" does not agree, because it does not allow directly to express the second member of the Boltzmann equation according to macroscopic quantities. Furthermore, we look here for an equation simplified to study Knudsen's layer near a wall ($K_n << 1$). The use of this method would mean so directly postulating solution. We will proceed in a different way. Here, the second method is used.

A Splitting of the Boltzmann equation

We first separate the collision operator of the Boltzmann equation in two parts. This step is very classical (see for instance [10]). Using the modelling (9) and introducing the functionals J_1 and J_2, the Boltzmann equation (8) may be written :

$$\frac{\partial f}{\partial t} + \vec{\xi}_1 \cdot \frac{\partial f}{\partial \vec{x}_1} + \vec{F} \cdot \frac{\partial f}{\partial \vec{\xi}_1} = \frac{1}{e^2}\left\{J_1 - e^2 J_2 f(\vec{\xi}_1, \vec{x}_1, t)\right\} \tag{16}$$

with :

$$J_1(\vec{\xi}_1, \vec{x}_1, t) = \frac{D^2}{e^2}\iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k})\left\{g_o(\vec{x}_1 + \frac{D}{2}\vec{k}, t) f(\vec{\xi}', \vec{x}_1, t) f(\vec{\xi}_2', \vec{x}_1 + D\vec{k}, t)\right\} d\vec{k} d\vec{\xi}_2 \tag{17}$$

and :

$$J_2(\vec{\xi}_1, \vec{x}_1, t) = \frac{D^2}{e^2}\iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k})\left\{g_o(\vec{x}_1 - \frac{D}{2}\vec{k}, t) f(\vec{\xi}_2, \vec{x}_1 - D\vec{k}, t)\right\} d\vec{k} d\vec{\xi}_2 \tag{18}$$

The first part (called J_1) is a distribution function after collision while the second ($J_2 f(\vec{\xi_1}, \vec{x_1}, t)$) is a distribution function before collision. An expression of these two functionnals is obtained thanks to an asymptotic expansion in terms of the small parameter $D/L << 1$. We obtain :

$$\begin{aligned} J_1(\vec{\xi}_1, \vec{x}_1, t) = {} & D^2 g_o(\vec{x}_1, t)\iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k}) f' f_2' d\vec{k} d\vec{\xi}_2 + D^3 g_o(\vec{x}_1, t)\iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k})\vec{k}\cdot f' \frac{\partial f_2'}{\partial \vec{x}_1} d\vec{k} d\vec{\xi}_2 \\ & + D^3 \frac{\partial g_o(\vec{x}_1, t)}{\partial \vec{x}_1} \cdot \iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k}) f' f_2' \vec{k} d\vec{k} d\vec{\xi}_2 + \mathcal{O}(D^4) \end{aligned} \tag{19}$$

and :

$$\begin{aligned} J_2(\vec{\xi}_1, \vec{x}_1, t) = {} & D^2 g_o(\vec{x}_1, t)\iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k}) f_2 d\vec{k} d\vec{\xi}_2 - D^3 g_o(\vec{x}_1, t)\iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k})\vec{k}\cdot \frac{\partial f_2}{\partial \vec{x}_1} d\vec{k} d\vec{\xi}_2 \\ & - D^3 \frac{\partial g_o(\vec{x}_1, t)}{\partial \vec{x}_1} \cdot \iint_{(\vec{g}\cdot\vec{k})>0} (\vec{g}\cdot\vec{k}) f_2 \vec{k} d\vec{k} d\vec{\xi}_2 + \mathcal{O}(D^4) \end{aligned} \tag{20}$$

Firstly, we work on the expression of $J_2(\vec{\xi_1}, \vec{x}_1, t)$.

B Modelling of $J_2(\vec{\xi_1}, \vec{x}_1, t)$

At this stage of the calculation, within the framework of the not linear models of the Boltzmann equation, assumptions are usually made to eliminate g of the integrals. It is case for instance, for the model BGK ([1], [10]). Indeed, this model is based on the use of a maxwellian potential which allows to modify slightly the writing of J_2. Introducing then the concept of "pseudo-maxwellian" particles, the authors make indirectly hypotheses on g. Then, this quantity disappears of the integrals and is absorbed by means of a constant frequency collision. In the case studied here, this kind of hypothesis is impossible to use. Indeed, the dynamics of the inelastic collision of two particles is not comparable with that of two molecules where attraction / repulsion forces play a dominating role. The same objections can be set against the use of a Lennard-Jones potential, often used in the kinetic theory of gas. In order to be consistent with the assumptions related to the collisions and supposing that particles are big enough to be able to neglect the interaction potential between particles and molecules of the carying fluid presented in [13], we can not use the pseudo-maxwellian particles assumption. Besides, with a rigid spheres potential, which is the most consistent with the retained collision modelling, the frequency of collision varies with the particles velocities, and this variation is expected to be important when the particles velocities are high ([3]). Objective is so to introduce in the model, a collision frequency depending on the particles velocities. By realizing integration in $\vec{k}$ in J_2, we obtain :

$$J_2(\vec{\xi}_1,\vec{x}_1,t) = D^2\left\{g_o(\vec{x}_1,t)\pi\int g f_2 d\vec{\xi}_2 - \frac{2\pi D}{3}g_o(\vec{x}_1,t)\int \vec{g}\cdot\frac{\partial f_2}{\partial \vec{x}_1}d\vec{\xi}_2 - \frac{2\pi D}{6}\frac{\partial g_o(\vec{x}_1,t)}{\partial \vec{x}_1}\cdot\int \vec{g} f_2 d\vec{\xi}_2\right\} + \mathcal{O}(D^4) \tag{21}$$

with $g = \| \vec{g} \|$. In the aim to obtain a simplified expression for J_2, we use a similar method to that of the usual case. We assume that $J_2(\vec{\xi}_1,\vec{x}_1,t)$ may be written under the form :

$$\begin{aligned} J_2(\vec{\xi}_1,\vec{x}_1,t) = D^2 \Big\{ & g_o(\vec{x}_1,t)\pi\theta(\vec{\xi}_1)\xi_1\int f_2 d\vec{\xi}_2 - \frac{2\pi D}{3}g_o(\vec{x}_1,t)\theta(\vec{\xi}_1)\vec{\xi}_1\cdot\frac{\partial}{\partial \vec{x}_1}\int f_2 d\vec{\xi}_2 \\ & - \frac{2\pi D}{6}\frac{\partial g_o(\vec{x}_1,t)}{\partial \vec{x}_1}\cdot\theta(\vec{\xi}_1)\vec{\xi}_1\int f_2 d\vec{\xi}_2\Big\} + \mathcal{O}(D^4) \qquad \text{with} \qquad \xi_1 = \| \vec{\xi}_1 \| \end{aligned} \tag{22}$$

and then :

$$\begin{aligned} J_2(\vec{\xi}_1,\vec{x}_1,t) = D^2 \Big\{ & g_o(\vec{x}_1,t)\pi\theta(\vec{\xi}_1)\xi_1 n(\vec{x}_1,t) - \frac{2\pi D}{3}g_o(\vec{x}_1,t)\theta(\vec{\xi}_1)\vec{\xi}_1\cdot\frac{\partial n(\vec{x}_1,t)}{\partial \vec{x}_1} \\ & - \frac{2\pi D}{6}\frac{\partial g_o(\vec{x}_1,t)}{\partial \vec{x}_1}\cdot\theta(\vec{\xi}_1)\vec{\xi}_1 n(\vec{x}_1,t)\Big\} + \mathcal{O}(D^4) \end{aligned} \tag{23}$$

That is, always with $D << L$:

$$J_2 = \pi\, D^2\xi_1\theta(\vec{\xi}_1)\left\{g_o(\vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{2\xi_1})n(\vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{\xi_1}) + \mathcal{O}(\bar{D}^2)\right\} \tag{24}$$

Then we set $N(\vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{\xi_1}) = g_o(\vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{2\xi_1})n(\vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{\xi_1})$ and taking this into account, J_2 is modelled by :

$$J_2 = \nu(\vec{x_1} - \frac{D\vec{\xi_1}}{\xi_1},t) \quad \text{with :} \quad \nu(\vec{x_1} - \frac{D\vec{\xi_1}}{\xi_1},t) = \pi\ D^2\ \xi_1\ \theta(\vec{\xi_1})\ g_o(\vec{x_1} - \frac{D\vec{\xi_1}}{3\xi_1},t)\ n(\vec{x_1} - \frac{2D\vec{\xi_1}}{3\xi_1},t) \tag{25}$$

where n is the number density and θ a function of the particle velocity.

C Full operator modelling

In accordance with the previous section, the new equation is :

$$\frac{\partial f}{\partial t} + \vec{\xi}_1\cdot\frac{\partial f}{\partial \vec{x}_1} + \vec{F}\cdot\frac{\partial f}{\partial \vec{\xi}_1} = \frac{1}{e^2}\left\{J_1 - e^2\nu(\vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{\xi_1})f(\vec{\xi}_1,\vec{x}_1,t)\right\} \tag{26}$$

As in the derivation of the usual BGK model, we put :

$$J_1 = \frac{1}{e^2}\ \nu(\vec{x_1} + \frac{D\vec{\xi_1}}{\xi_1},t)\ f_o(\vec{\xi_1},\vec{x_1},t) \tag{27}$$

where $f_o(\vec{\xi_1},\vec{x_1},t)$ is a Maxwellian distribution function. This choice expresses the tendency of the suspension to return towards an equilibrium state. It is confirmed by the existence of an H theorem for the original Boltzmann equation. To satisfy in best the properties of the original collision operator, it is advisable to bring the biggest care to the choice of this function. It is chosen as the maxwellian satisfying the following three integral properties:

▷ Two properties allowing the conservation of mass and momentum :

$$\int_{\vec{\xi}_1}\nu(\vec{\xi}_1,\vec{x}_1 + \frac{\bar{D}\vec{\xi}_1}{\xi_1},t)f_o(\vec{\xi}_1,\vec{x}_1,t)\psi d\vec{\xi}_1 = \int_{\vec{\xi}_1}e^2\nu(\vec{\xi}_1,\vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{\xi_1},t)f(\vec{\xi}_1,\vec{x}_1,t)\psi d\vec{\xi}_1 \tag{28}$$

with $\psi = \vec{\xi}_1$ and $\psi = 1$.

▷ One assumption about the energy balance :

$$\int_{\vec{\xi}_1} \nu(\vec{\xi}_1, \vec{x}_1, t) f_o(\vec{\xi}_1, \vec{x}_1, t) {\xi_1}^2 d\vec{\xi}_1 = \int_{\vec{\xi}_1} \nu(\vec{\xi}_1, \vec{x}_1, t) f(\vec{\xi}_1, \vec{x}_1, t) {\xi_1}^2 d\vec{\xi}_1 \tag{29}$$

The density, velocity and energy which appear in $f_o(\vec{\xi}_1, \vec{x}_1, t)$ are not the five local moments, but some fictitious local parameters related to the five moments of f weighted with $\nu(\vec{\xi}_1, \vec{x}_1, \vec{\xi}_1)$. The obtained BGK model is therefore :

$$\frac{\partial f}{\partial t} + \vec{\xi}_1 \cdot \frac{\partial f}{\partial \vec{x}_1} + \vec{F} \cdot \frac{\partial f}{\partial \vec{\xi}_1} = \frac{1}{e^2} \left\{ \nu(\vec{\xi}_1, \vec{x}_1 + \frac{\bar{D}\vec{\xi}_1}{\xi_1}, t) f_o(\vec{\xi}_1, \vec{x}_1, t) - e^2 \nu(\vec{\xi}_1, \vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{\xi_1}, t) f(\vec{\xi}_1, \vec{x}_1, t) \right\} \tag{30}$$

This model is similar to the usual BGK model in the case of non punctual and inelastic collisions of identical hard spheres.

One can show that by adding some assumptions to this model, one finds known models. Supposing that collisisons are punctual and elastic ($e = 1$) and supposing that the shield effect is negligible ($g_o = 1$), the equation (30) may be written :

$$\frac{\partial f}{\partial t} + \vec{\xi}_1 \cdot \frac{\partial f}{\partial \vec{x}_1} + \vec{F} \cdot \frac{\partial f}{\partial \vec{\xi}_1} = \nu(\vec{\xi}_1, \vec{x}_1, \vec{\xi}_1) \left\{ f_o(\vec{\xi}_1, \vec{x}_1, t) - f(\vec{\xi}_1, \vec{x}_1, t) \right\} \tag{31}$$

with :

$$\nu(\vec{\xi}_1, \vec{x}_1, t) = \frac{9\pi \bar{D}^2}{4} \xi_1 \theta(\vec{\xi}_1) n(\vec{x}_1, t) \tag{32}$$

It is a generalization of the BGK model with a collision frequency which depends on the particles velocity ([3], [11]). The previous relation is a particular expression of the collision frequency ν.

Furthermore, if the assumption of pseudo-maxwellian particles is made, the collision frequency does not depend any more on the particles velocity and it mays be written :

$$\nu(\vec{\xi}_1, \vec{x}_1, t) = A\, n(\vec{x}_1, t) \tag{33}$$

where the value of A may be found in [10] for instance. The obtained model is then the usual BGK model.

III PROPERTIES OF THE BGK MODEL EQUATION

A Equation of moments

In a very classical way in kinetic theory, the equation of moments is written :

$$\int \left\{ \frac{\partial f}{\partial t} + \vec{\xi}_1 \cdot \frac{\partial f}{\partial \vec{x}_1} + \vec{F} \cdot \frac{\partial f}{\partial \vec{\xi}_1} \right\} \psi d\vec{\xi}_1 = \mathcal{C}(\psi) \tag{34}$$

with :

$$\mathcal{C}(\psi) = \frac{1}{e^2} \int \left\{ \nu(\vec{\xi}_1, \vec{x}_1 + \frac{\bar{D}\vec{\xi}_1}{\xi_1}, t) f_o(\vec{\xi}_1, \vec{x}_1, t) - e^2 \nu(\vec{\xi}_1, \vec{x}_1 - \frac{\bar{D}\vec{\xi}_1}{\xi_1}, t) f(\vec{\xi}_1, \vec{x}_1, t) \right\} \psi d\vec{\xi}_1 \tag{35}$$

Due to the assumptions (28), the conservation of mass ($\psi = 1$) and of momentum ($\psi = \vec{\xi}_1$) are automatically verified. On the other hand, as for the original operator, this equation does not give the conservation of energy ($\psi = \xi^2$). Model supplies so, at this level, the same type of properties as the original equation except for the momentum. However, if the assumption (which we interhook mainly here) of small particles is made, the BGK equation and the Boltzmann equation have the same properties. Under this assumption, it remains:

$$\mathcal{C}(\psi) = \frac{1}{e^2}\int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - e^2\nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\}\psi d\vec{\xi_1}$$
$$+ \frac{D}{e^2}\int \frac{\vec{\xi_1}}{\xi_1}\cdot\vec{\nabla}\nu(\vec{\xi_1},\vec{x}_1,t)\psi\left(f_o(\vec{\xi_1},\vec{x}_1,t) + e^2 f(\vec{\xi_1},\vec{x}_1,t)\right)d\vec{\xi_1} + \mathcal{O}(\bar{D}^2) \tag{36}$$

B *H* Theorem

The functions $\mathcal{H}(\vec{x_1},t)$ and $\vec{\phi}_H(\vec{x_1},t)$ are defined in the same way as in the first section by the relations (12) and (13) respectively. In the same way as in the first section, the following law of balance for the $\mathcal{H}$ quantity is obtained :

$$\frac{\partial}{\partial t}\mathcal{H}(\vec{x_1},t) + \frac{\partial}{\partial \vec{x_1}}\cdot\vec{\phi}_H(\vec{x_1},t) = \frac{1}{e^2}\int \left\{\nu(\vec{\xi_1},\vec{x}_1 + \frac{\bar{D}\vec{\xi_1}}{\xi_1},t)f_o(\vec{\xi_1},\vec{x}_1,t)\right.$$
$$\left. - e^2\nu(\vec{\xi_1},\vec{x}_1 - \frac{\bar{D}\vec{\xi_1}}{\xi_1},t)f(\vec{\xi_1},\vec{x}_1,t)\right\}[1 + ln(f)]\,d\vec{\xi_1} \tag{37}$$

In the case of small particles and using the conservation of mass , it remains :

$$\frac{\partial}{\partial t}\mathcal{H}(\vec{x_1},t) + \frac{\partial}{\partial \vec{x_1}}\cdot\vec{\phi}_H(\vec{x_1},t) = \frac{1}{e^2}\int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - e^2\nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\} ln\left(\frac{f}{f_o}\right)d\vec{\xi_1}$$
$$+ \frac{1}{e^2}\int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - e^2\nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\} ln(f_o)d\vec{\xi_1}$$
$$+ \mathcal{O}(\bar{D}) \tag{38}$$

In the cas of weakly inelastic collisions, we set $e = 1 - \eta$ with $\eta << 1$. Consequently, the previous expression is reduced to :

$$\frac{\partial}{\partial t}\mathcal{H}(\vec{x_1},t) + \frac{\partial}{\partial \vec{x_1}}\cdot\vec{\phi}_H(\vec{x_1},t) = \int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - \nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\} ln\left(\frac{f}{f_o}\right)d\vec{\xi_1}$$
$$+ \int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - \nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\} ln(f_o)d\vec{\xi_1}$$
$$+ \mathcal{O}(\bar{D}) + \mathcal{O}(\eta) \tag{39}$$

The function f_o being maxwellian, $ln(f_o)$ may be written according to the quantites 1, $\vec{\xi_1}$ and ${\xi_1}^2$. Consequently, the second inegral of the previous expression may be written:

$$\int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - \nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\} ln(f_o)d\vec{\xi_1}$$
$$= \int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - \nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\}\left[\alpha_o + \vec{\alpha_1}\cdot\vec{\xi_1} + \alpha_2{\xi_1}^2\right]d\vec{\xi_1} \tag{40}$$

By using the assumption (29), it remains, in the second integral of the previous expression, only the terms "1" and "$\vec{\xi_1}$", which nullify by using a linearization of the first order of relations (28):

$$\int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - \nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\} ln(f_o)d\vec{\xi_1} = 0 + \mathcal{O}(\bar{D}) \tag{41}$$

Consequently, the balance equation for $\mathcal{H}(\vec{x_1},t)$ is reduced to :

$$\frac{\partial}{\partial t}\mathcal{H}(\vec{x_1},t) + \frac{\partial}{\partial \vec{x_1}}\cdot\vec{\phi}_H(\vec{x_1},t) = \int \left\{\nu(\vec{\xi_1},\vec{x}_1,t)f_o(\vec{\xi_1},\vec{x}_1,t) - \nu(\vec{\xi_1},\vec{x}_1,t)f(\vec{\xi_1},\vec{x}_1,t)\right\} ln\left(\frac{f}{f_o}\right)d\vec{\xi_1}$$
$$+ \mathcal{O}(\bar{D}) + \mathcal{O}(\eta) \tag{42}$$

The collision frequency is necessarily greater than 0 and the term :

$$\left\{ f_o(\vec{\xi}_1, \vec{x}_1, t) - f(\vec{\xi}_1, \vec{x}_1, t) \right\} ln \left(\frac{f(\vec{\xi}_1, \vec{x}_1, t)}{f_o(\vec{\xi}_1, \vec{x}_1, t)} \right) \tag{43}$$

is always lower than 0. We have therefore at the first order :

$$\frac{\partial}{\partial t} \mathcal{H}(\vec{x_1}, t) + \frac{\partial}{\partial \vec{x_1}} \cdot \vec{\phi}_H(\vec{x_1}, t) \leq 0 \tag{44}$$

Consequently, if we consider small particles and weakly inelastic collisions, we have an H theorem for the BGK model, that is the same result as for the original Boltzmann equation.

IV CONCLUSION

The problem of a two-phase dispersed medium is studied within the kinetic theory. In the case of small and undeformable spheres that have all the same radius, a model for the collision operator of the Boltzmann equation is proposed. The collisions are supposed instantaneous, binary and inelastic. The obtained collision operator allows, to prove the existence of an H theorem in the case of a suspension of small hard spheres having binaray, instantaneous and weakly inelastic collisions. Because of the complexity of the non linear structure of the collision integral of the Boltzmann equation, it is interesting to introduce a BGK model equation. All the basic formal properties of the original operator, are retained by this model equation in the case of small particles. Consequently, this model is consistent with the original kinetic description and it will be very usefull in the study of some complicated problems like the Knudsen layer for instance.

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